Optimal Trajectory Planning with Dynamic Constraints for Autonomous Vehicle
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Abstract: Trajectory planning with high accuracy is one of the key technologies of autonomous vehicles to travel safely on public roads. In this paper, we investigate the dependence of the path and velocity interactive planning based on cubic polynomial with vehicle dynamic as well as kinematical constraints including lateral acceleration and road curvature. The effect due to the vehicle dynamics of the host car by ignoring dependence states is studied through the comparison of longitudinal acceleration data in a simulation study and on-road test. Results show that our proposed methodology is more safety and comfort than the one without considering the dependency.

Keywords: autonomous vehicle; lateral acceleration; trajectory planning; vehicle dynamic.

1. INTRODUCTION

Autonomous vehicles (AVs) have great potential to improve active traffic safety and give human being a convenient life. Since the 1980s, the intelligence of AVs has increased from simply closed field to complex on-road testing field [1]-[5]. The architecture of the AVs can be illustrated by the following general architecture in Fig. 1, including map library, perception, mission planning, and behavior planning. A static-global route based on a user-set destination is first planned from mission planning [6]-[7], and then the motion planning subsystem calculates the local trajectory from the current state to the next local target state defined by the behavior planning subsystem. Moreover, the local trajectory from the motion planning subsystem satisfies vehicle dynamic constraints, and can avoid collisions with any static and moving obstacle around the AVs.

![Fig.1 A general control structure for autonomous driving system.](image)

A key problem to solve before the AVs become a product is how to stably determine a collision-free path and velocity of the AVs in terms of local trajectory planning. Moreover, for the trajectory planning, the polynomial curves are one of the common methods in AVs application, interpolates a target point and establishes a smooth path with velocity based on a polynomial function along with vehicle dynamic and kinematical constraints. [8]-[9] applied curvature polynomials with third degree to ensure continuous rate of change of curvature with a straightforward numerical procedure in real time. After that, McNaughton et al. [10] proposed a planner that first samples target points in behavior planning along with the road using curvature polynomials to satisfy all road shape. Trajectories generated in [11] implemented fourth degree polynomials for lane change scenarios with longitudinal constraints, and further studied fifth degree polynomials for the lateral constraints. Besides, for overtaking scenarios, cubic polynomials were used in [12] to generate optimal trajectories based on cost function. A few research platforms, however, have shown great capability in driving on public roads due to planning longitudinal and lateral polynomials independently, where these methods cannot be applied on a complex on-road driving.

In this paper, we propose a dependent trajectory planning for both velocity and path based on cubic polynomials combining vehicle kinematic as well as dynamic constraints to plan a safe and comfortable trajectory. Moreover, the Karush-Kuhn-Tucker (KKT) method [13] is applied for nonlinear programming constraints including lateral acceleration and road curvature. To study the effect of dependent longitudinal and lateral plans as ignoring the relationship between path and velocity, both numerical experiment at PreScan and on-road testing are considered. The results of the both tests showed that the proposed model provides more suitable trajectories with lower lateral acceleration than the one with independent planning for path and velocity.

The rest of the paper is organized as follows. Section 2 presents the architecture of proposed system design, and optimal trajectory planning in linking the host...
vehicle dynamic constrains is described in Section 3. Both simulation study and on-road testing for evaluating our methodology are provided in Section 4, and Section 5 is the concluding remarks.

2. SYSTEM FRAMEWORK

The proposed autonomous driving system (ADS) constructed by the three subsystems: candidate target point selection subsystem, real-time motion decision subsystem and tracking control subsystem. The overall system framework is shown in Fig. 2.

Fig.2 The framework of the ADS which can optimize path and velocity trajectory simultaneously.

**Behavior Generator Subsystem:**
In order to decide where to go in the next moment, a good ADS should have the ability that automatically detects its surroundings including the current position, the obstacles position and lane stripes. Thus, we need to select the optimal trajectory. The main function of the candidate target selection subsystem is to select a set of the candidate target points from map data and submits them to real-time motion decision subsystem for selecting the optimal target point. It extracts the necessary information from the environment of the autonomous vehicle via RTK-GPS sensors, map data and fusion sensors.

**Real-Time Motion Planning Subsystem:**
The real-time motion decision subsystem can be seen as the brain of the AV. It can generate the optimal path and the optimal velocity curve for the AV based on its surroundings. It consists of path trajectory generator and velocity trajectory generator and cost function analysis unit. The main function of real-time motion decision subsystem is to select the optimal path trajectory and the optimal velocity trajectory. The real-time motion decision employs the cost function to measure the effect of the surrounding environment to AV. These cost functions are constructed based on the knowledge of structural road and defensive driving criterions. By calculating the summation of each cost function, the real-time motion decision subsystem can determine the optimal trajectory.

**Tracking Control Subsystem:**
When the optimal trajectory is obtained, the tracking controller will take over it and guides the autonomous vehicle to the target point.

3. OPTIMAL TRAJECTORY PLANNING

In this study, we focus on the constrained vehicle trajectory optimization problem. Since the surrounding environments of the AV change at any time, the vehicle dynamics will severely limit the behavior of the AV. Unlike the traditional trajectory optimization problem, this study focuses on the optimal trajectory planning problem with the vehicle dynamic constraints.

3.1 Vehicle trajectory optimization problem

The proposed trajectory planner is based on the following vehicle kinematical model [8]:

\[
\begin{align*}
\dot{x}_t &= v_t \cos(\theta_t) \\
\dot{y}_t &= v_t \sin(\theta_t) \\
\dot{\theta}_t &= \kappa_t \\
\dot{\kappa}_t &= \delta_t
\end{align*}
\]

where \( (x_t, y_t) \) denote the center mass of the AV with respect to the origin point \( o \) of the map data and \( \theta_t \) denotes the heading angle with respect to \( o \); the term \( v_t \) denotes the velocity of the center of the AV; \( \delta_t \) denote the curvature control signal.

Assume the velocity of the center of the AV \( v_t \) and curvature control signal \( \delta_t \) can be respectively represented as:

\[ v_t = v(s, q) \triangleq q_3 s^3 + q_2 s^2 + q_1 s + q_0 \]

and

\[ \delta_t = \delta(s, p) \triangleq 3p_3 s^2 + 2p_2 s + p_1 \]

where \( q = [q_3, q_2, q_1, q_0]^T \) and \( p = [p_3, p_2, p_1]^T \).

The vehicle kinematical model in Eq. (1) can be rewritten as:

\[
\begin{align*}
\dot{x}(s, p) &= \cos(\theta(s, p)) \\
\dot{y}(s, p) &= \sin(\theta(s, p)) \\
\dot{\theta}(s, p) &= \kappa(s, p) \\
\dot{\kappa}(s, p) &= \delta(s, p)
\end{align*}
\]

Thus, the path trajectory \( x(s, p) \) of the AV with arc length \( s \) can be defined as:

\[
x(s, p) = [x(s, p) \ y(s, p) \ \theta(s, p) \ \kappa(s, p)]^T
\]

where

\[
\begin{align*}
x(s, p) &= \int_0^s \cos(\theta(r, p))dr \\
y(s, p) &= \int_0^s \sin(\theta(r, p))dr \\
\theta(s, p) &= \int_0^s \kappa(r, p)dr \\
\kappa(s, p) &= \int_0^s 3p_3 r^2 + 2p_2 r + p_1 dr
\end{align*}
\]

By appropriately selecting \( v(s, q) \) and \( \delta(s, p) \), we can efficiently lead the AV to a desired target vector \( \eta_d = [x_d^T, v_d^T]^T \).

Thus, for a given target vector \( \eta_d \), the optimal trajectory can be obtained by solving the following optimization problem:

\[
\min f(s_f, \zeta) = \int_0^{s_f} (\eta(r, \zeta) - \eta_d)^T(\eta(r, \zeta) - \eta_d) dr
\]

subject to

\[
\eta(s_f, \zeta) - \eta_d = 0
\]

where \( \zeta = [p^T, q^T]^T \) and \( \eta(r, \zeta) = [x(r, p)^T, v(r, q)]^T \).
3.2 Dynamic constrained vehicle trajectory optimization problem

In this subsection, we discuss the effect of vehicle dynamics to the vehicle trajectory including acceleration, deceleration and lateral acceleration.

In general, the range of the acceleration $a_t$ of a vehicle should be limited in $[a, \bar{a}]$ where $a$ denotes the maximum deceleration and $\bar{a}$ denotes the maximum acceleration.

$$ a \leq a_t = \frac{dv}{dt} = (3q_3s^2 + 2q_2s + q_1)v(s, q) \leq \bar{a}. $$

The acceleration and deceleration constraint can be respectively rewritten as follows:

$$ v(s, q) \leq \frac{\bar{a}}{3q_3s^2 + 2q_2s + q_1}, $$

$$ v(s, q) \geq \frac{a}{(3q_3s^2 + 2q_2s + q_1)}. $$

For increasing the level of passenger comfort, we also need to consider the effect of lateral acceleration. In general, the lateral acceleration $a_{lat}$ should be bound in $\bar{a}_{lat}$. Thus, the lateral acceleration constraint can be defined as:

$$ a_{lat} = \frac{v^2}{\kappa(s, p)} = \frac{v(s, q)^2}{\kappa(s, p)} \leq \bar{a}_{lat}. $$

where $\bar{a}_{lat}$ denotes the maximum lateral acceleration.

By introducing the constraints in (6)-(8), the dynamic constrained vehicle trajectory optimization problem for a given target vector $\eta_d$ can be defined as follows:

$$ \min_{s, f, \zeta} J(s_f, \zeta) = \int_0^{s_f} (\eta(r, \zeta) - \eta_d)^T(\eta(r, \zeta) - \eta_d) \, dt $$

subject to

$$ \eta(s_f, \zeta) - \eta_d = 0, $$

$$ (3q_3s^2 + 2q_2s + q_1)v(s, q) - \bar{a} \leq 0, $$

$$ a - (3q_3s^2 + 2q_2s + q_1)v(s, q) \leq 0, $$

$$ v(s, q)^2 - a_{lat}\kappa(s, p) \leq 0. $$

Set Lagrangian to be

$$ \mathcal{L}(z, \lambda, \mu) = f(z) + \lambda^T T(z) + \mu^T h(z). $$

The corresponding KKT conditions would be:

$$ \nabla \mathcal{L}(z, \lambda, \mu) = 0 $$

$$ T(z) = 0 $$

$$ h_j(z) \leq 0 $$

$$ \mu_j \geq 0 $$

$$ \mu_j(z)h_j(z) = 0 $$

where

$$ z = [s_f, \zeta]^T $$

$$ T(z) = \eta(s_f, \zeta) - \eta_d $$

$$ h_1 = (3q_3s^2 + 2q_2s + q_1)v(s, q) - \bar{a}, $$

$$ h_2 = a - (3q_3s^2 + 2q_2s + q_1)v(s, q), $$

and $h_3 = v(s, q)^2 - \bar{a}_{lat}\kappa(s, p).$

By solving the roots of (11)-(15), we can obtain the optimal trajectory satisfied with the vehicle dynamic constraints. Fortunately, the roots of (11)-(15) can be efficiently solved by using optimization toolbox of the commercial software MATLAB.

3.3 Generating optimal trajectory based on cost function analysis

Based on above analysis, one can obtain the optimal vehicle dynamic constrained trajectory of the given target vector $\eta_d = [x_d, v_d]^T$. However, in practical application, the AV employs behavior generator subsystem to generate a set of candidate target vector called the candidate target vector set $\Omega \triangleq \{\eta_d\}$. Then, real-time motion planning subsystem of the AV will select the optimal target vector $\eta_{d^*}$ from the candidate target vector set $\Omega$. A problem appears that for a given candidate target vector set $\Omega = \{\eta_d\}$ which candidate target vector $\eta_d$ will be the optimal target vector $\eta_{d^*}$.

In this study, we employ cost function analysis to decide which target vector $\eta_d$ is optimal. Thus, the optimal target vector for a given candidate target vector set $\Omega = \{\eta_d\}$ can be represented as:

$$ \min_{\eta_d \in \Omega} C(\eta_d) = \sum_{i=1}^{n} w_i c_i(\eta_d) $$

where $c_i(\cdot)$ denote the $i$th cost function; the term $w_i$ denote the weighting with respect to the $i$th cost function. The adopted cost functions are given in Table-I.

<table>
<thead>
<tr>
<th>Cost</th>
<th>Formula</th>
<th>Physical Interpretation</th>
<th>Weight</th>
<th>Impact</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c_1$</td>
<td>$\xi$</td>
<td>$\xi$ is path length</td>
<td>$w_1 = 1$</td>
<td>efficiency</td>
</tr>
<tr>
<td>$c_2$</td>
<td>$\sum_{i=1}^{n} k_i$</td>
<td>$k_i$ is curvature</td>
<td>$w_2 = 10$</td>
<td>comfort</td>
</tr>
<tr>
<td>$c_3$</td>
<td>$\sum_{i=1}^{n} k_i$</td>
<td>$k_i$ is rate of change of curvature</td>
<td>$w_3 = 10$</td>
<td>comfort</td>
</tr>
<tr>
<td>$c_4$</td>
<td>$\sum_{i=0}^{n} a_i$</td>
<td>$a_i$ is lateral offset with the closest center line</td>
<td>$w_4 = 10$</td>
<td>behavior</td>
</tr>
<tr>
<td>$c_5$</td>
<td>$d_{ab}$</td>
<td>$d_{ab}$ is distance between the host car and obstacle</td>
<td>$w_5 = 1$</td>
<td>safety</td>
</tr>
<tr>
<td>$c_6$</td>
<td>$t$</td>
<td>$t$ is time duration of a trajectory</td>
<td>$w_6 = 10$</td>
<td>efficiency</td>
</tr>
<tr>
<td>$c_7$</td>
<td>$\sum_{i=0}^{n} v_i^2$</td>
<td>$v_i$ is speed</td>
<td>$w_7 = 1$</td>
<td>energy</td>
</tr>
<tr>
<td>$c_8$</td>
<td>$\sum_{i=0}^{n} a_i^2$</td>
<td>$a_i$ is acceleration</td>
<td>$w_8 = 0.1$</td>
<td>comfort</td>
</tr>
<tr>
<td>$c_9$</td>
<td>$\sum_{i=0}^{n} l_i^2$</td>
<td>$l_i$ is jerk (the rate of change of acceleration)</td>
<td>$w_9 = 0.1$</td>
<td>comfort</td>
</tr>
<tr>
<td>$c_{10}$</td>
<td>$\sum_{i=0}^{n} v_i^2k_i$</td>
<td>$k_i$ is centripetal acceleration</td>
<td>$w_{10} = 0.1$</td>
<td>comfort</td>
</tr>
</tbody>
</table>

Once the optimal trajectory is obtained, the vehicle dynamic control unit will take over and guide the AV to the optimal target vector. The weights values are adjustable for different driving behaviors, but we are not starting to focus this part. So far, the cost functions method can be used to design driving behaviors, but the only way is trying and tuning. We design a lot of
coefficients to make all combinations that could happen on real roads.

4. EXPERIMENTAL RESULTS

The path tracking algorithm is connected optimal trajectory calculated from real-time motion decision module and EPS controller that determines the overall performance result. In this experiment, developing pure pursuit algorithm to send command to the EPS controller for controlling the AV moving as close as possible the path that designs from the trajectory planning module. Before doing on-road testing, it has to design three controllers for EPS motor, throttle and brake actuators, in order to control the steering angle rotating as quickly as possible from the steering wheel angle command, and control two actuators executing as soon as possible from throttle and brake command. After calculating optimal trajectory from motion planner module, we have the path and velocity plan that are ready to be sent to path tracker algorithm and velocity tracker algorithm. In addition, in our experiment, the values of $\ddot{x}$, $\dot{a}$ and $\ddot{a}^{\text{lat}}$ are set to be $4\text{m/s}^2$, $-8\text{m/s}^2$ and $-3\text{m/s}^2$, respectively.

4.1 PreScan simulation analysis

In this paper, the proposal solution is evaluated by this novel approach through PreScan, which is a physics-based simulation platform that is used in the automotive industry for development. In order to ensure safety and performance of this approach, building the standard testing scenario that is a big loop like a sport field with three lanes for doing lane changing and obstacle avoiding. Setting a millimeter wave radar on the front bumper to detect cars ahead, the AV gets relative distance and velocity from cars in front. Setting localization module, the AV gets its position and heading among the map library. In Fig. 3, the AV detects a car stopped in front, and plans optimal trajectory immediately. This scenario is standard for testing optimal trajectory planning with dynamic constraints in PreScan.

![Fig.3 The standard testing scenario in the PreScan.](image)

According to the absolute position and velocity of the surrounding objects, the ADS is going to process behavior planner and motion planner in real-time. Finally, the simulation result shows that the AV can slow down to stop behind the car ahead as Fig. 4(a) and can change to the slide lane to avoid the stopped car ahead as Fig. 4(c). With the defensive mode, the result as Fig. 4(a) and Fig. 4(b) shows that the red line is the path and velocity of the motion planner and the AV decided to slow down until stopping behind the static car. Because the cost function is designed with specific coefficient for becoming defensive driving behavior. In another case as Fig. 4(c) and Fig. 4(d), the AV chooses more aggressive mode to engage lane changing to avoid

![Fig.4 the path and velocity planning in defensive mode (a), (b) and aggressive mode (c), (d).](image)
collision and reduce its speed when executing the lane change.

4.2 Implementation and experiment on vehicle

At next step, we are ready to implement the approach on real vehicle, and the platform is a SUV car equipped with a 3D LiDAR mounted on the front and a RTK-GPS set on the roof as Fig. 5 shown below. The map library provides the centerline of two lanes of the test scene, and it is important information for doing implementation. The motion planning module is then tested on this experiment vehicle.

![Fig.5 The experiment vehicle for on-road testing.](image)

As the scenario built in PreScan, the experiment is set an electric golf carts as a leading car which is easy to be detected by 3D LiDAR system. The object detection system integrated 3D LiDAR with RTK-GPS localization system to transfer the position of the objects from relative distance to earth-fixed coordinate. We designed the cost function, and we adjusted weightings of the cost function to make the AV wants to do lane change rather than be slow down to follow the leading car. According to the target point selection module, it is going to do trajectory planning with the start point and the target point which represent the current position and final position as Fig. 6 shown below.

![Fig.6 The on-road testing standard scenario.](image)

After planning an optimal trajectory with the aggressive mode, the AV finds out the optimal trajectory planning with dynamic constraints with the result like the simulation in PreScan as Fig. 4(c) and Fig. 4(d). But, the different part is that using optimal trajectory with vehicle dynamic constraints causes the AV gets lower lateral acceleration than the approach without vehicle dynamic constraints In Fig. 7 shown below, the novel approach with Lagrange multiplier designed shows that the AV could choose the optimal trajectory with lower lateral acceleration and passengers could feel more comfort. The dynamic constraints are limited on $+0.3g$ to $−0.3g$ as the two red horizontal red lines.

![Fig.7 The lateral acceleration comparison with Lagrange multiplier being constraints.](image)

Therefore, among all trajectories that can be selected, after being lateral acceleration restricted, the AV will select the trajectory with the minimum lateral acceleration one.

The traditional method may avoid the obstacle as well as new method, but it has higher lateral acceleration than the traditional method. The figure 7 shows the result in two different methods. The maximum lateral acceleration is $2.2 \text{ m/s}^2$ in our new method and the worse performance in the traditional method is $4.8 \text{ m/s}^2$. To plan the better trajectory, these two new and traditional trajectories are not the same length in time domain; therefore, the result shows the traditional one costs longer time than new one and implies that the value in the same time cannot be compared because two trajectories are not the same lengths. The only one we sure is the maximum acceleration is smaller than the traditional one.

5. CONCLUSION

In this paper, we proposed a motion planning module with cost function makes optimal trajectory optimization with a practical real-time platform. The trajectory optimization method is based on the KKT conditions created a smooth curve with safety lateral acceleration in both simulated and on-road demonstrations. Besides, results further showed the decisions between changing lanes, avoiding obstacles, and slowing down are difficult to keep distance with leading cars when the AVs are driving on busy road.

Our proposed methodology for solving the dependence between path and velocity constraints is important for the autonomous vehicle industry faced with the problem that each system plan is mutually dependent. The velocity and path, obviously, affect each other and further affect vehicle safety as well as comfort.
Although the cost function independently combines the physical quantities, the interaction between each other is bound to be the future work.

Last but not least, the target point selection method in Fig. 2 plays a key technology for where the safety target point chooses from current state with an efficient methodology for the issue of iteration consume.

REFERENCES